EPFL - Autumn semester 2025-2026 J. Scherer
Cohomology rings **Solution 4**
MATHEMATICS

Exercice 1. The transfer map. Let $\alpha\colon H \hookrightarrow G$ be a subgroup and M a $\mathbb{Z}G$-module (which can be viewed as a $\mathbb{Z}H$-module). We continue the particular case of Sheet 3, Exercise 5, Point 5.

1. When H has finite index in G, show that $\operatorname{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, M) \cong \mathbb{Z}G \otimes_{\mathbb{Z}H} M$ as $\mathbb{Z}G$-modules.
2. When H has finite index in G, compute the composite
$$M \to \operatorname{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, M) \cong \mathbb{Z}G \otimes_{\mathbb{Z}H} M \to M$$
3. When H has finite index in G, compute $H_*(G; \operatorname{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, M))$ and deduce from this the existence of homomorphisms $\operatorname{tr}_H^G\colon H_*(G; M) \to H_*(H; M)$. Dualize briefly.
4. Show that the composition $\alpha_* \circ \operatorname{tr}_H^G$ is multiplication by the index. We only ask for the homological case here.
5. When G is a finite group, prove that $|G| \cdot H_k(G; M) = 0$ for $k > 0$ and any $\mathbb{Z}G$-module M.
6. Under the same assumptions as in Point 5, conclude that if $|G|$ is invertible in M, then $H_k(G; M) = 0$ for $k > 0$.

Solution 1

1. The module $\mathbb{Z}G$ is a free $\mathbb{Z}H$-module, so for any $\mathbb{Z}H$-module M, there is a natural isomorphism :
$$\operatorname{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, M) \to \mathbb{Z}G \otimes_{\mathbb{Z}H} M, \quad f \mapsto \sum_i g_i \otimes f(g_i^{-1}),$$
where $\{g_i\}$ is a set of left coset representatives for H in G. This map is $\mathbb{Z}G$-linear and has an inverse given by defining $g \otimes m \mapsto (x \mapsto f(x) = xg^{-1} \cdot m)$, extended linearly.
2. Consider the composite :
$$M \to \operatorname{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, M) \to \mathbb{Z}G \otimes_{\mathbb{Z}H} M \to M,$$
where :
   - — The first map sends $m \in M$ to the $\mathbb{Z}H$-linear map $f_m \in \operatorname{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, M)$ defined by $f_m(g) = g \cdot m$ (from Sheet 3, Exercise 5)
   - — The second map is the isomorphism from (1).
   - — The third map is defined on simple tensors by $g \otimes m \mapsto g \cdot m$.

   Applying all three steps to $m \in M$, we get :
$$m \mapsto f_m \mapsto \sum_i g_i \otimes g_i^{-1} \cdot m \mapsto \sum_i g_i \cdot (g_i^{-1} \cdot m) = \sum_i m = [G : H]m.$$
3. Assume that $F_\bullet \longrightarrow \mathbb{Z} \longrightarrow 0$ is a free resolution of the trivial G-module. Then $H_*(G, Hom_{\mathbb{Z}H}(\mathbb{Z}G, M))$ is the homology of
$$F_\bullet \otimes_{\mathbb{Z}G} Hom_{\mathbb{Z}H}(\mathbb{Z}G, M) \longrightarrow 0.$$
But for each i we have :

$$F_i \otimes_{\mathbb{Z}G} (\mathbb{Z}G \otimes_{\mathbb{Z}H} M) \cong (F_i \otimes_{\mathbb{Z}G} \mathbb{Z}G) \otimes_{\mathbb{Z}H} M \cong F_i \otimes_{\mathbb{Z}H} M$$

Now note that each free $\mathbb{Z}G$ module is a free $\mathbb{Z}H$ module so $F_\bullet$ is also a free resolution of the trivial $\mathbb{Z}H$-module, so the homology of $F_\bullet \otimes_{\mathbb{Z}H} M \longrightarrow 0$ is $H_*(H, M)$. So $H_*(G, Hom_{\mathbb{Z}H}(\mathbb{Z}G, M)) \cong H_*(H, M)$.

Moreover, we obtain a map from $H_i(G, M)$ to $H_i(G, Hom_{\mathbb{Z}H}(\mathbb{Z}G, M))$, applying $Tor_i(\mathbb{Z}, -)$ to the first map from (2) : $M \to Hom_{\mathbb{Z}H}(\mathbb{Z}G, M))$. And since $H_i(G, Hom_{\mathbb{Z}H}(\mathbb{Z}G, M))$ is isomorphic to $H_i(H, M)$, we have the desired trace map.

4. Let $F_\bullet$ be a resolution of $\mathbb{Z}$ by $\mathbb{Z}G$-modules. Then, $F_\bullet$ gives us also a resolution of $\mathbb{Z}$ by $\mathbb{Z}H$-modules). Now,
$$\alpha_* : H_*(H; M) \to H_*(G; M)$$
is induced by
$$\tilde{\alpha} : F_\bullet \otimes_{\mathbb{Z}H} M \to F_\bullet \otimes_{\mathbb{Z}G} M$$
given by $f_n h^{-1} \otimes m = f_n \otimes h \cdot m \mapsto f_n \alpha(h)^{-1} \otimes m = f_n \otimes \alpha(h) \cdot m$. Notice that $\tilde{\alpha}\colon F_\bullet \otimes_{\mathbb{Z}G} \mathbb{Z}G \otimes_{\mathbb{Z}H} M \to F_\bullet \otimes_{\mathbb{Z}G} M$ is precisely induced by the canonical evaluation map
$$\mathbb{Z}G \otimes_{\mathbb{Z}H} M \to M.$$
Putting together with (2) and (3) we get that $\alpha_* \circ \mathrm{tr}_H^G$ is just $\cdot[G : H]$.
5. Since G is finite we have that $|G/H| = |G|/|H|$. From (4),
$$|G| \cdot H_k(G; M) = |H| \cdot |G/H| \cdot H_k(G; M) =$$
$$= |H| \cdot (\alpha_* \circ \mathrm{tr}_H^G)(H_k(G; M)).$$
Since $\alpha_* \circ \mathrm{tr}_H^G$ factors through $H_k(H; M)$ we might as well pick $H = \{1\}$. Then the index $|G/H| = |G|$ so we get that
$$|G| \cdot H_k(G; M) = (\alpha_* \circ \mathrm{tr}_{\{1\}}^G)(H_k(G; M))$$
and since $H_k(\{1\}, M) = 0$ for all $k > 0$ we have that
$$(\alpha_* \circ \mathrm{tr}_{\{1\}}^G)(H_k(G; M))$$
factors through 0, so we then get
$$|G| \cdot H_k(G; M) = 0$$
for all $k > 0$.
6. From (5), $|G| \cdot H_k(G; M) = 0$ and if $|G|$ is invertible we can multiply both sides of the equation by its inverse, obtaining the desired equality.

♢ **Exercice 2. The norm map.** Let C_n be a cyclic group of order n, and t a generator. Let M be a left $\mathbb{Z}C_n$-module and $N\colon M \to M$ be the multiplication by the sum $1 + t + \cdots + t^{n-1}$ of all elements in C_n.

1. Identify the cochain complex $\mathrm{Hom}_{\mathbb{Z}C_n}(F_\bullet, M)$ where $F_\bullet$ is the periodic resolution from Week 1 (the map N appears there!).

2. Show that N induces a map $\overline{N}\colon M_{C_n} \to M^{C_n}$, called the *norm map.*
3. Show that $H^{2n}(C_n; M) \cong M^G/(N \cdot M) \cong \operatorname{Coker} \overline{N}$ for $n \geq 1$.
4. Show that all odd cohomology groups are isomorphic to the kernel of $\overline{N}$.
5. Identify all cohomology groups of C_n with trivial coefficients $\mathbb{Z}$ and $\mathbb{F}_p$.
6. Prove that $\operatorname{Coker} \overline{N}$ and $\ker \overline{N}$ are annihilated by n (use a map $M^{C_n} \to M_{C_n}$).

Solution 2. (by Yvane and Marko) In this exercise, we introduce the norm map and use it to give explicit descriptions of the cohomology groups.

1. First of all, recall the following periodic resolution of $\mathbb{Z}$ as a $\mathbb{Z}C_n$-module :

$$F_\bullet : \quad \cdots \to \mathbb{Z}C_n \xrightarrow{N} \mathbb{Z}C_n \xrightarrow{t-1} \mathbb{Z}C_n \xrightarrow{N} \mathbb{Z}C_n \xrightarrow{t-1} \mathbb{Z}C_n \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

   Let us identify the cochain complex $\operatorname{Hom}_{\mathbb{Z}C_n}(F_\bullet, M)$. Observe that any $\phi \in \operatorname{Hom}_{\mathbb{Z}C_n}(\mathbb{Z}C_n, M)$ is uniquely determined by $\phi(1) \in M$. Therefore, $\operatorname{Hom}_{\mathbb{Z}C_n}(\mathbb{Z}C_n, M) \simeq M$. Moreover, by definition, $\operatorname{Hom}_{\mathbb{Z}C_n}(t-1, M)$ is also the multiplication by $(t-1)$. Indeed :

$$\operatorname{Hom}_{\mathbb{Z}C_n}(t-1, M) : \phi \in \operatorname{Hom}_{\mathbb{Z}C_n}(\mathbb{Z}C_n, M) \mapsto \Big(1 \mapsto \phi(t-1) = (t-1)\cdot\phi(1) \in M\Big) = (t-1)\cdot\phi$$

   By the same argument, $\operatorname{Hom}_{\mathbb{Z}C_n}(N, M)$ is also the multiplication by $1+t+\ldots+t^{n-1}$. Therefore,

$$\operatorname{Hom}_{\mathbb{Z}C_n}(F_\bullet, M) : \quad 0 \to M \xrightarrow{t-1} M \xrightarrow{N} M \xrightarrow{t-1} M \xrightarrow{N} M \to \cdots$$

2. We now show that N induces a map $\bar{N} : M_{C_n} \to M^{C_n}$ called the *norm map.* First, observe that for any $m \in M$, $\quad t \cdot N(m) = (t+\ldots+t^{n-1}+t^n)\cdot m = (1+t+\ldots+t^{n-1})\cdot m = N(m)$, i.e. the image of N is contained in the invariants of M so that the corestriction $N : M \to M^{C_n}$ is well-defined. Moreover, $N(t \cdot m - m) = 0$ for any $m \in M$. Hence, by the universal property of the quotient $M_{C_n} \simeq M/ < t\cdot m - m >$, N induces a map $\bar{N} : M_{C_n} \to M^{C_n}$.
3. We claim that $H^{2n}(C_n; M) \simeq \operatorname{coker}\bar{N}$ for $n \geq 1$. By definition of cohomology groups, $H^{2n}(C_n; M) = \ker(t-1)/\operatorname{im}N$ and since $\operatorname{im}N = \operatorname{im}\bar{N}$ and $\ker(t-1) = M^{C_n}$, the desired result follows.
4. We show that $H^{2n+1}(C_n; M) \simeq \ker\bar{N}$ for $n \geq 0$. By definition of cohomology groups, $H^{2n+1}(C_n; M) = \ker(N)/\operatorname{im}(t-1)$. Denote by $q : M \to M_{C_n}$ the quotient map. Then by construction of coinvariants, $\ker q = \operatorname{im}(t-1)$ and since $N = \bar{N} \circ q$, the quotient map induces an isomorphism $\ker\bar{N} \simeq \ker N/\ker q = H^{2n+1}(C_n; M)$.
5. Our next goal is to identify all cohomology groups with trivial coefficients $\mathbb{Z}$ and $\mathbb{F}_p$. First, we observe that, if C_n acts trivially on M then $M_{C_n} \simeq M \simeq M^{C_n}$ and in particular $H^0(C_n; M) \simeq M^{C_n} \simeq M$. Moreover, for any $m \in M$, $\bar{N}(m) = N(m) = n \cdot m$ so that $\bar{N}$ is just the multiplication by n.
   Now, if $M = \mathbb{Z}$, we obtain

$$H^k(C_n; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } k = 0 \\ 0 & \text{if } k \text{ is odd} \\ \mathbb{Z}/n\mathbb{Z} & \text{if } k > 0 \text{ is even} \end{cases}$$

   If $M = \mathbb{F}_p$ and if p does not divide n, then the multiplication by n is an isomorphism so that $H^0(C_n; \mathbb{F}_p) = \mathbb{F}_p$ and all higher cohomology groups are trivial. In the case where p divides n, then the multiplication by n is the zero map so that $\ker N = \mathbb{F}_p = \operatorname{coker} N$. Therefore $H^k(C_n; \mathbb{F}_p) = \mathbb{F}_p$ for all k.

6. Finally, we prove that coker$\bar{N}$ and ker$\bar{N}$ are annihilated by n. Denote by $q' = q|_{M^{C_n}}$ the restriction of the quotient map. For $m \in M^{C_n}$, we have $\bar{N} \circ q'(m) = N(m) = n \cdot m$ since m is invariant. Therefore, $n \cdot m$ is in the image of $\bar{N}$ and becomes 0 in its cokernel i.e. n annihilates coker$\bar{N}$.
   Let $[m] \in M_{C_n}$ then $q' \circ \bar{N}([m]) = [N(m)] = n \cdot [m]$. Therefore, if $m \in$ker$\bar{N}$ then $n \cdot [m] = q' \circ \bar{N}([m]) = q'(0) = 0$ and ker$\bar{N}$ is also annihilated by n.

♢ **Exercice 3. Shapiro Lemma.** Let $H < G$ be a subgroup, M be a left $\mathbb{Z}H$-module and define the induced module $\mathrm{Ind}_H^G M = \mathbb{Z}G \otimes_{\mathbb{Z}H} M$ and the coinduced module $\mathrm{Coind}_H^G M = \mathrm{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, M)$, where $\mathbb{Z}G$ is seen as right or left module over $\mathbb{Z}H$ via group multiplication.

1. Prove that $\mathrm{Ind}_H^G M$ contains M as a submodule over $\mathbb{Z}H$ and decomposes as a direct sum of the gM's where g ranges over a set of representatives for G/H.
2. Let M be the permutation module $\mathbb{Z}[G/H]$, where G acts by left multiplication on the cosets. Show that it is isomorphic to $\mathrm{Ind}_H^G \mathbb{Z}$.
3. Prove that $H_n(H; M) \cong H_n(G; \mathrm{Ind}_H^G M)$.
4. Prove that $H^n(H; M) \cong H^n(G; \mathrm{Coind}_H^G M)$.
5. Show that induced modules from the trivial subgroup $1 < G$ are acyclic in homology.

Solution 3

1. Let $e_G \otimes M := \{e_G \otimes m \mid m \in M\}$. It is a $\mathbb{Z}H$-submodule of $\mathrm{Ind}_H^G M$. Indeed, for $h \in H$

   $$e_G \otimes m + h \cdot (e_G \otimes m') = e_G \otimes m + h \otimes m' = e_G \otimes m + e_G \otimes (h \cdot m') = e_G \otimes (m + h \cdot m') \in e_G \otimes M.$$

   And it is isomorphic to M (as a $\mathbb{Z}H$-module). Now observe that for each left-coset $gH \in {}^{G}\!/_{H}$, we have a $\mathbb{Z}H$-submodule $g \otimes M = gH \otimes M$ in $\mathrm{Ind}_H^G M$. In particular, $H \otimes M = e_G \otimes M$. We prove that these are submodules in the same way :

   $$g \otimes m + h \cdot (g \otimes m') = g \otimes m + gh \otimes m' = g \otimes m + g \otimes (h \cdot m') = g \otimes (m + h \cdot m') \in gH \otimes M.$$

   As the action of g on $e_G \otimes M$ is given by $g \cdot (e_G \otimes m) = g \otimes m$, we can see that $e_G \otimes M \approx M$ is mapped (isomorphically) to $g \otimes M$ by the action of g. So we have $gH \otimes M = gM$.
   Let $E \subset G$ be a set of one representative of each coset. Since any such set (freely) generates $\mathbb{Z}G$ as a $\mathbb{Z}H$-module, we have

   $$\mathrm{Ind}_H^G M = \sum_{g \in E} gH \otimes M = \sum_{g \in E} gM \tag{1}$$

   Furthermore, we have that for $g \notin H, m \neq 0$, $g \otimes m$ cannot be put under the form $e_G \otimes m'$, so

   $$g \otimes M \cap e_G \otimes M = 0_{\mathrm{Ind}_H^G M}.$$

   This implies that for $g, g' \in G$ such that $gH \neq g'H$,

   $$g \otimes M \cap g' \otimes M = g'^{-1} g \otimes M \cap e_G \otimes M = 0_{\mathrm{Ind}_H^G M}$$

   So the sum in 1 is a direct sum of abelian groups, and hence a direct sum of $\mathbb{Z}G$-modules.

   $$\mathrm{Ind}_H^G M = \bigoplus_{g \in E} gM = \bigoplus_{\bar{g} \in {}^{G}\!/_{H}} \bar{g}M$$

2. By question 1, we know that

$$\mathrm{Ind}_H^G\mathbb{Z} = \bigoplus_{\bar g \in {}^{G}\!/_{H}} \bar g\mathbb{Z}$$

There is an obvious isomorphism

$$\mathbb{Z}[{}^{G}\!/_{H}] \approx \bigoplus_{\bar g \in {}^{G}\!/_{H}} \bar g\mathbb{Z}$$

But let us give an explicit isomorphism. We can write a general element of $\mathrm{Ind}_H^G\mathbb{Z}$ under the form

$$(\sum_{x\in G} n_x\ x) \otimes m = (\sum_{x\in E}\sum_{h\in H} n_{xh}\ xh) \otimes m = \sum_{x\in E}(x \sum_{h\in H} n_{xh}\ h \otimes m) = \sum_{x\in E}(x \otimes \sum_{h\in H} n_{xh}\ h \cdot m)$$

and, as the action of H on $\mathbb{Z}$ is trivial,

$$= \sum_{x\in E}(x \otimes m \sum_{h\in H} n_{xh}) = \sum_{x\in E}(n_{\bar x} x \otimes m) = (\sum_{x\in E} n_{\bar x} x) \otimes m$$

where we set $n_{\bar x} = \sum_{h\in H} n_{xh}$. So we can give the explicit $\mathbb{Z}G$-isomorphism :

$$\mathbb{Z}[{}^{G}\!/_{H}] \longleftrightarrow \mathrm{Ind}_H^G\mathbb{Z}$$

$$\sum_{\bar x \in {}^{G}\!/_{H}} n_{\bar x}\ \bar x \longrightarrow \sum_{x\in E}(x \otimes n_{\bar x}) = (\sum_{x\in E} n_{\bar x} x) \otimes 1$$

$$m \sum_{\bar x \in {}^{G}\!/_{H}} n_{\bar x} \bar x \longleftarrow (\sum_{x\in G} n_x x) \otimes m = (\sum_{x\in E} n_{\bar x} x) \otimes m$$

3. First, let us show that we have that for any G-module N and H-module M the isomorphism

$$N \otimes_{\mathbb{Z}H} M = N \otimes_{\mathbb{Z}G} (\mathbb{Z}G \otimes_{\mathbb{Z}H} M) \tag{2}$$

where N is considered as an H-module on the left (through restriction of scalars). Or, written under an other form

$$(N \otimes_{\mathbb{Z}} M)_H = (N \otimes_{\mathbb{Z}} (\mathbb{Z}G \otimes_{\mathbb{Z}} M)_H)_G$$

This directly follows from the properties of the tensor product. On the right, we have element of the form $n \otimes g \otimes m$ with the relations (for $h \in H, g \in G$)

$$n \otimes h \otimes m = n \otimes e_G \otimes h \cdot m$$

$$gn \otimes g \otimes m = gn \otimes g \cdot (e_G \otimes m) = n \otimes e_G \otimes m$$

and on the left, we have element of the form $n \otimes m$ with the relations (for $h \in H$)

$$h \cdot n \otimes h \cdot m = n \otimes m$$

and so clearly the mapping

$$n \otimes m \leftrightarrow n \otimes e_G \otimes m$$

is an isomorphism, since the relation on the right are respected on the left (and vice versa)

$$h \cdot n \otimes h \cdot m \mapsto h \cdot n \otimes e_G \otimes h \cdot m = h \cdot n \otimes h \otimes m = n \otimes e_G \otimes m$$

Let F be a projective $\mathbb{Z}G$-resolution of $\mathbb{Z}$. We can look at F as a projective resolution over $\mathbb{Z}H$, again by restriction of scalars, so applying the definition yields

$$\begin{tikzcd}
H_*(F \otimes_{\mathbb{Z}H} M) \arrow[r, equal, "2"] \arrow[d, equal, "def"] & H_*(F \otimes_{\mathbb{Z}G} (\mathbb{Z}G \otimes_{\mathbb{Z}H} M)) \arrow[d, equal, "def"'] \\
H_*(H; M) \arrow[r, equal] & H_*(G, \mathbb{Z}G \otimes_{\mathbb{Z}H} M)
\end{tikzcd}$$

where the equality on the bottom is what we wanted to prove

$$H_*(H; M) = H_*(G; \mathrm{Ind}_H^G M).$$

4. By Brown's book (III.3.6), the co-extension of scalars functor is right-adjoint to the restriction of scalars functor. This means that for any G-module N and H-module M we have

$$Hom_G(N, \mathrm{Hom}_H(\mathbb{Z}G, M)) = Hom_H(N, M). \tag{3}$$

This isomorphism is proven in the book of Milne.
Let $\alpha : N \to \mathrm{CoInd}_H^G M$ be a G-homomorphism (we write $\alpha_m := \alpha(m)$). For each $m \in N$, we have a diagram

$$\begin{tikzcd}
e_G \arrow[r, mapsto] \arrow[d, mapsto, "\alpha_m"] & m \arrow[dl, dashed, "\beta"'] \\
M &
\end{tikzcd}$$

and to construct $\beta : N \to M$ so that the diagram commutes, it is clear that we have to set

$$\beta(m) := \alpha_m(e_G)$$

Now taking into account the G-equivariance of α

$$\beta(g \cdot m) = \alpha_{g \cdot m}(e_G) = (g \cdot \alpha)_m(e_G) = \alpha_m(e_G \cdot g) = \alpha_m(g)$$

where we used the definition of the action of G on the coinduced module in the last step. And as α_m is an H-homomorphism, if $g \in H$:

$$\alpha_m(g) = g \cdot \alpha_m(e_G) = g \cdot \beta(m)$$

so β is indeed a H-homomorphism ($\beta \in Hom_H(N, M)$).
Now let $\beta : N \to M$ be an H-homomorphism. We define α so that for $m \in N$, $g \in G$

$$\alpha_m(g) := \beta(g \cdot m)$$

$$\begin{tikzcd}
e_G \arrow[r, mapsto] \arrow[dr, dashed, "\alpha_m"'] & g \cdot m \arrow[d, mapsto, "\beta"] \\
& M
\end{tikzcd}$$

and α is a G-homomorphism :

$$\alpha_{g \cdot m}(g') = \beta(g' \cdot (g \cdot m)) = \beta(g'g \cdot m) = \alpha_m(g'g) = (g \cdot \alpha)_m(g').$$

And those two maps are inverses of each other, so we have the desired isomorphism and we have proven the adjunction.

Now let F be a projective $\mathbb{Z}G$-resolution of $\mathbb{Z}$. By applying the definition of group cohomology with coefficients in a module, we have

$$\begin{tikzcd}
H^*(Hom_H(F,M)) \arrow[r, equal, "3"] \arrow[d, equal, "def"] & H^*(Hom_G(F,\mathrm{Hom}_H(\mathbb{Z}G,M)) \arrow[d, equal, "def"'] \\
H^*(H;M) \arrow[r, equal] & H^*(G,\mathrm{Hom}_H(\mathbb{Z}G,M))
\end{tikzcd}$$

where the equality on the left is true because we can look at F as a $\mathbb{Z}H$-resolution of $\mathbb{Z}$ (by restriction of scalars), and the equality on the top comes from the adjunction. The equality on the bottom is what we wanted to prove

$$H^*(H;M) = H^*(G;\mathrm{CoInd}_H^G M).$$

5. Let us recall that $\mathrm{Ind}^G M$ is acyclic in homology if for all $n \geq 0$, $H_n(G;\mathrm{Ind}^G M) = 0$.
   By using the first part of the Shapiro lemma (question 3), we have that
   $$H_*(G;\mathrm{Ind}^G M) = H_*(1;M)$$
   So $\mathrm{Ind}^G M$ is acyclic (as a G-module) if and only if M is acyclic (as a 1-module).

   By applying the definition,
   $$H_*(1,M) = H_*(F \otimes_{\mathbb{Z}} M)$$
   where F is a free resolution of $\mathbb{Z}1 = \mathbb{Z}$. We can take
   $$F:\ 0 \longrightarrow \mathbb{Z} \xrightarrow{\varepsilon = \mathrm{id}_{\mathbb{Z}}} \mathbb{Z} \longrightarrow 0$$
   and then we have
   $$F \otimes_{\mathbb{Z}} M:\ 0 \longrightarrow \mathbb{Z} \otimes_{\mathbb{Z}} M \xrightarrow{\varepsilon \otimes \mathrm{id}_M + \mathrm{id}_{\mathbb{Z}} \otimes 0} \mathbb{Z} \otimes_{\mathbb{Z}} M \longrightarrow 0$$
   or simply (as $\varepsilon \otimes \mathrm{id}_M + \mathrm{id}_{\mathbb{Z}} \otimes 0 = \mathrm{id}_{\mathbb{Z} \otimes_{\mathbb{Z}} M}$ and $\mathbb{Z} \otimes_{\mathbb{Z}} M \approxeq M$)
   $$0 \longrightarrow M \xrightarrow{\mathrm{id}_M} M \longrightarrow 0$$
   which is clearly exact, and so all of its homology group are trivial. Therefore, as wanted, $H_n(1,M) = 0$ for all $n \geq 0$, and the induced module is acyclic in homology.

◊ **Exercice 4. Semi-direct products.** Let G be a group, fix a $\mathbb{Z}G$-module M, and consider an extension $0 \to M \xrightarrow{i} E \xrightarrow{\pi} G \to 1$.

1. Show that the extension splits if and only if there is a subgroup $H < E$ such that $E = i(M) \cdot H$ and $i(M) \cap H = \{1\}$.
2. Show that the extension splits if and only if there is a subgroup $H < E$ such that every element of E can be expressed uniquely as $e = i(m)h$ for $m \in M, h \in H$.
3. Show that the extension splits if and only if it is equivalent to the semi-direct product extension $M \rtimes G$.

4. Show that the symmetric group S_3 is a semi-direct product $\mathbb{Z}/3 \rtimes C_2$. Compute $H^1(C_2; \mathbb{Z}/3)$ for the corresponding $\mathbb{Z}C_2$-module structure on $\mathbb{Z}/3$ and find all conjugacy classes of sections of the extension $\mathbb{Z}/3 \to S_3 \to C_2$.
5. Same exercice as point 4 for the product $\mathbb{Z}/2 \times C_2$.
6. **Bonus** : Same exercise as point 4 for the unitriangular matrix group in $GL_3(\mathbb{F}_3)$ as a semi-direct product $(\mathbb{Z}/3 \times \mathbb{Z}/3) \rtimes C_3$. This is the subgroup of upper triangular matrices with 1's on the diagonal. It has order 27 and all elements except 1 have order 3 (prove this).

Solution 4.(by Sandro and Patrick)

1. "$\Rightarrow$" : Assume that the extension splits, i.e. the morphism π admits a section $s : G \to E$. As s is a group morphism, $H := s(G)$ is a subgroup of E. For any $e \in E$ we can write

$$e = e \cdot \big(s\pi(e)\big)^{-1} \cdot s\pi(e) = \Big(e \cdot \big(s\pi(e)\big)^{-1}\Big) \cdot \big(s\pi(e)\big)$$

with $s\pi(e) \in H$ by definition and

$$\pi\Big(e \cdot \big(s\pi(e)\big)^{-1}\Big) = \pi(e) \cdot \pi\Big(s\pi(e)\Big)^{-1} = \pi(e)\big(\pi(e)\big)^{-1} = 1,$$

hence $e \cdot \big(s\pi(e)\big)^{-1} \in \ker(\pi) = i(M)$. Thus, $E = i(M)H$.

Now, let $e \in i(M) \cap H$. Then, $\pi(x) = 1$, since $x \in i(M) = \ker(\pi)$. As $x \in H$, there exists $g \in G$, such that $x = s(g)$. Then

$$1 = \pi(x) = \pi\big(s(g)\big) = g,$$

hence we deduce that $x = s(1) = 1$. Thus, $i(M) \cap E = \{1\}$.

"$\Leftarrow$" : We know that $\pi|_H$ is injective. Indeed $\ker(\pi|_H) = \ker(\pi) \cap H = i(M) \cap H = \{1\}$. Further it is surjective. Indeed let $g \in G$ and pick $e \in E$ such that $\pi(e) = g$. Write $e = i(m)h$ for $m \in M$ and $h \in H$. Then,

$$\pi(h) = \pi\big(i(m)\big) \cdot \pi(h) = \pi\big(i(m)h\big) = \pi(e) = g.$$

Thus, $\pi|_H$ is an isomorphism. Let $s : G \to H \subseteq E$ be its inverse postcomposed with the inclusion of H into E. Then, by construction s is a section.

2. "$\Rightarrow$" : Take $H = s(G)$ as in part 1. We know, that $E = i(M)H$ and $i(M) \cap H = \{1\}$. Let $i(m)h = i(m')h'$. Then

$$i(-m + m') = hh'^{-1} \in i(M) \cap H = \{1\}.$$

In particular, $m = m'$, since i is injective and $h = h'$. So the decomposition is unique.

"$\Leftarrow$" : We will show that $E = i(M)H$ and $i(M) \cap H = \{1\}$ such that we can conclude by part 1. Clearly $E = i(M)H$, as each element in E admits a decomposition in a product of an element in $i(M)$ and an element in H. Let $x \in i(M) \cap H$. Then, x admits two decompositions : $x \cdot 1$ and $i(1) \cdot x$, where we view x once as an element in $i(M)$ and once as an element in H. By uniqueness of the expression as a product, they have to coincide, and thus $x = 1$. Hence $i(M) \cap H = \{1\}$ and we conclude.

3. We are going to prove the equivalence by interchangeably using the formulations of split extensions given by part 1 and 2.
   "⇒" Consider the following diagram

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "i"] \arrow[d, equal] & E \arrow[r, "\pi"] \arrow[d, "\psi"] & G \arrow[r] \arrow[d, equal] & 1 \\
0 \arrow[r] & M \arrow[r, "\iota"] & M \rtimes G \arrow[r, "p"] & G \arrow[r] & 1,
\end{tikzcd}$$

   where $\psi : E \to M \rtimes G$ is defined by $\psi(e) = \big(m, \pi(e)\big)$, where $e = i(m)s(g)$ is the unique expression given by the hypothesis and part 2, i.e. $m \in M$ and $s(g) \in H = s(G)$. Remark that $\pi(e) = g$. By construction this map lets the above diagram commute. We have to show, why ψ is a group homomorphism. For this we have to analyse, how we can write $e \cdot e' = i(m)s(g)i(m')s(g')$ as a product of an element of $i(M)$ and an element in H. We have

$$i(m)s(g)i(m')s(g') = i(m)s(g)i(m')s(g)^{-1}s(gg') = i(m)i(gm')s(gg') = i(m + gm')s(gg'),$$

   since the action of G on M is precisely the pulling back via i of the conjugation in E of $i(m)$ by a preimage of g. Hence, $\psi(ee') = (m + gm', gg') = \psi(e)\psi(e')$, so ψ is a group morphism. The only thing left to prove is that ψ is an isomorphism. Let $(m, g) \in M \rtimes G$, then consider $e = i(m)s(g) \in E$. By construction $\psi(e) = (m, g)$. Finally, ψ is injective by uniqueness of the expression $e = i(m)s(g)$ and since s is injective, beeing a right inverse of π. All in all, ψ is an isomorphism which makes the above diagram commute, so we are done.
   "⇐" : Suppose the existence of an isomorphism ψ such that the diagram above commutes. We know that the lower row comes equipped with a section given by the inclusion

$$s \colon G \cong 0 \rtimes G \hookrightarrow M \rtimes G : g \mapsto (0, g).$$

   This is a group morphism since $s(gg') = (0, gg') = (0, g) \cdot (0, g') = s(g) \cdot s(g')$. Moreover, clearly $p \circ s = id_G$. Then $\psi^{-1} \circ s$ yields a section of the above extension by commutativity of the diagram, so we are done.
4. Consider $K = \langle (123) \rangle \cong \mathbb{Z}/3 \trianglelefteq S_3$. Then we have a short exact sequence

$$0 \to K \hookrightarrow S_3 \to S_3/K \cong C_2 \to 0.$$

   Note that $K \cong \mathbb{Z}/3$ is a $S_3/K \cong C_2$-module letting the generator of C_2 acts as the automorphism of K that swaps the two non-identity elements. We can see this, as the action has to correspond to the conjugation in S_3 by a preimage of the generator. We can choose the preimage (12), and hence deduce this module structure.
   Moreover, the above sequence admits a section $s : C_2 \to S_3$ that sends the generator to the transposition (12). Therefore by part 3. we have an isomorphism $S_3 \cong \mathbb{Z}/3 \rtimes C_2$. By Exercise 2 of this exercise sheet

$$H^1(C_2, \mathbb{Z}/3) \cong \ker(\overline{N})$$

   with $\overline{N} : (\mathbb{Z}/3)_{C_2} \to (\mathbb{Z}/3)^{C_2}$ the norm map. But here $(\mathbb{Z}/3)_{C_2} \cong \{1\}$, since in the submodule by which we quotient, we have $t \cdot 1 - 1 = 2 - 1 = 1$ and $t \cdot 2 - 2 = 1 - 2 = 2$, where t denotes the generator of C_2. Hence we quotient by all of $\mathbb{Z}/3$, so $(\mathbb{Z}/3)_{C_2}$ is trivial. We deduce, that

$$H^1(C_2, \mathbb{Z}/3) = 0.$$

   By the correspondence seen in class, there is thus only one section of the extension up to conjugation. In particular, the three section given by sending t to a transposition are all equivalent up to conjugation.

5. We consider the short exact sequence

$$0 \to \mathbb{Z}/2 \to \mathbb{Z}/2 \times C_2 \to C_2 \to 0,$$

where the maps are given by the inclusion in the first factor and the projection onto the second factor. Since the group $\mathbb{Z}/2 \times C_2$ is abelian, the action of C_2 on $Z/2$ is trivial, since any conjugation in $\mathbb{Z}/2 \times C_2$ is just the identity. Again using Exercise 2 of this sheet,

$$H^1(C_2, \mathbb{Z}/2) \cong \ker(\overline{N}),$$

where $\overline{N} : (\mathbb{Z}/2)_{C_2} \to (\mathbb{Z}/2)^{C_2}$ is the norm map. Since the action of C_2 on $\mathbb{Z}/2$ is trivial, we deduce that $(\mathbb{Z}/2)_{C_2} = (\mathbb{Z}/2)^{C_2} = \mathbb{Z}/2$. Furthermore, the map $\overline{N}$ is just multiplication by 2. Hence $\ker(\overline{N}) = \mathbb{Z}/2$. Hence, there are two sections of the extension

$$0 \to \mathbb{Z}/2 \to \mathbb{Z}/2 \times C_2 \to C_2 \to 0$$

up to conjugation.

In fact, there are only two group homomorphisms $C_2 \to \mathbb{Z}/2 \times C_2$ that send the generator of C_2 to itself in the second component of the product, namely the diagonal map and the inclusion in the second factor, so the above tells us that these are not conjugate and represent the only two possibilities.

6. Note that this group is given by

$$UT(3, \mathbb{F}_3) = \{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} | a, b, c \in \mathbb{F}_3 \}.$$

We can freely choose a, b, c to be any element of $\mathbb{F}_3$, so the order of the group is $3^3 = 27$. Moreover, for any element

$$A = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \in UT(3, \mathbb{F}_3)$$

we have

$$A^2 = \begin{pmatrix} 1 & 2a & 2b + ac \\ 0 & 1 & 2c \\ 0 & 0 & 1 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 1 & 3a & 3b + 3ac \\ 0 & 1 & 3c \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

so all nontrivial elements have order 3, as desired. Let M be the subgroup of all $A \in UT(3, \mathbb{F}_3)$ such that $A_{23} = 0$. It has order $3^2 = 9$, hence by the fact that all elements have order 3, it must be isomorphic to $\mathbb{Z}/3 \times \mathbb{Z}/3$. Consider the short exact sequence

$$0 \to \mathbb{Z}/3 \times \mathbb{Z}/3 \hookrightarrow UT(3, \mathbb{F}_3) \xrightarrow{p} UT(3, \mathbb{F}_3)/(\mathbb{Z}/3 \times \mathbb{Z}/3) =: G \to 0.$$

Since G has order 3 it is isomorphic to C_3.

Any nontrivial element of G is a generator, choose for example the class of

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

To specify a section of p it suffices to specify the image $s([B])$ of this generator so that $p(s([B])) = [B]$. We can trivially choose $s([B]) = B$, so the sequence splits, meaning that $UT(3, \mathbb{F}_3) \cong (\mathbb{Z}/3 \times \mathbb{Z}/3) \rtimes C_3$. Notice that the action of G on M is given by $[B] \cdot (1,0) = (1,2), [B] \cdot (0,1) = (0,1)$, where we have identified the first and second copy of $\mathbb{Z}/3$ in the product with

$$\begin{pmatrix} 1 & * & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 1 & 0 & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{respectively.}$$

Indeed, we may take B as a lift of $[B]$ in $UT(3, \mathbb{F}_3)$ and hence

$$[B] \cdot \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = B \cdot \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot B^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The second calculation is similar.
In particular, $M_G \cong \mathbb{Z}/3$ since $\langle \{g \cdot m - m | g \in G, m \in M\} \rangle = 0 \times \mathbb{Z}/3$ and $M^G = 0 \times \mathbb{Z}/3 \cong \mathbb{Z}/3$. Moreover, the map $\overline{N}$ is induced by the action of $1+[B]+[B]^2$. It maps the representative [(1,0)] on $[(1,0) + (1,2) + \big((1,2) + (0,2)\big)] = [(0,0)]$. Hence, $\overline{N}$ is the zero map, and we deduce by Exercise 2

$$H^1(C_3, \mathbb{Z}/3 \times \mathbb{Z}/3) \cong \ker(\overline{N}) \cong \mathbb{Z}/3.$$

As before, this means the above extension admits 3 sections up to conjugation. We can find a total of 9 possible sections, which are, as discussed above, uniquely determined by the image of $[B]$. We will sort them in their respective conjugacy classes. The three conjugacy classes are

$$\left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \right\},$$

$$\left\{ \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \right\},$$

$$\left\{ \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

◊ **Exercice 5. Hochschild cohomology.** Let k be a commutative ring, A a (unital) k-algebra with multiplication μ, and M an A-bimodule (the left and right actions commute : $a(mb) = (am)b$). We write $\otimes$ for the tensor product of k-modules. Recall Hochschild homology from exercise 5, Week 1.

1. Define A^{op} as the *opposite* k-algebra where the product $a \cdot b = ba$. We write $A^e = A \otimes A^{\mathrm{op}}$. Show that a left A^{op}-module is the same as a right A-module, and thus an A^e-module is an A-bimodule.
2. Define the bar complex $C^{bar}_\bullet(A)$ with $C_n(A) = A^{\otimes n+2}$ (with augmentation given by μ). Show that $HH_n(A; M)$ is the homology of the chain complex $M \otimes_{A^e} C^{bar}_\bullet(A)$.
3. Define $HH^n(A, M)$ as the cohomology of the cochain complex $\mathrm{Hom}(C^{bar}_\bullet(A), M)$. Identify this with the cohomology of $\mathrm{Hom}_k(A^\bullet, M)$ (write down the differentials of this cochain complex).
4. Identify $HH^0(A; M)$ with the invariants M^A (define them).

5. Identify $HH^1(A;M)$ with the quotient of the k-module of derivations $Der(A,M)$ by the submodule of inner derivations (and define them).

Solution 5. (by Louis and Léo)

1. We write $A^e = A \otimes A^{\mathrm{op}}$. We show that a A^e module is an A-bimodule. One familiar with these notions could recall the definition of left (resp. right) A-module as an additive functor

$$\mathrm{A} \text{ (resp. } \mathrm{A}^{\mathrm{op}}) \to \mathrm{Ab}$$

where A (resp. A^{op}) is the single-object category with the elements of A (resp. A^{op}) as morphisms, and composition of morphisms being multiplication in A (resp. A^{op}). As the definition directly explicits it, a left A-module is the same as a right A^{op}-module (since $(\bullet^{\mathrm{op}})^{\mathrm{op}} = \bullet$). We also prove it with less categorical background : let M be a left A^{op}-module and write $\cdot_{\mathrm{op}}$ for both the multiplication in A^{op} and the module action by A^{op}. Then, M is also a right A-module by defining

$$M \times A : (m,a) \to m \cdot a := a \cdot_{\mathrm{op}} m \in M.$$

Indeed, the four axioms of being a right A-module are easily verified. They shouldn't be a problem but we still write the calculations :

   (a) $(m+n) \cdot a = a \cdot_{\mathrm{op}} (m+n) = a \cdot_{\mathrm{op}} m + a \cdot_{\mathrm{op}} n = m \cdot a + n \cdot a$
   (b) $m \cdot (a+b) = (a+b) \cdot_{\mathrm{op}} m = a \cdot_{\mathrm{op}} m + b \cdot_{\mathrm{op}} m = (a+b) \cdot_{\mathrm{op}} m = m \cdot (a+b)$
   (c) $m \cdot (a \cdot b) = (b \cdot_{\mathrm{op}} a) \cdot_{\mathrm{op}} m = b \cdot_{\mathrm{op}} (a \cdot_{\mathrm{op}} m) = (m \cdot a) \cdot b$
   (d) $m \cdot 1 = 1 \cdot_{\mathrm{op}} m = m$

Now, let M be a left $A^e = A \otimes A^{\mathrm{op}}$-module. Using the ring homomorphisms

$$\begin{aligned} \iota_L : A &\to A \otimes A^{\mathrm{op}} \\ a &\mapsto a \otimes 1 \end{aligned} \qquad\qquad \begin{aligned} \iota_R : A^{\mathrm{op}} &\to A \otimes A^{\mathrm{op}} \\ a &\mapsto 1 \otimes a \end{aligned}$$

We have that M is an left A-module and a left A^{op}-module (i.e. a right A-module). Moreover the two actions are compatible since for all $a, b \in A$ and all $m \in M$,

$$(a \cdot m) \cdot b = ((a \otimes 1) \cdot m) \cdot b = (1 \otimes b) \cdot ((1 \otimes 1) \cdot m) = ((1 \otimes b)(a \otimes 1)) \cdot m = (a \otimes b) \cdot m$$

$$a \cdot (m \cdot b) = a \cdot ((1 \otimes b) \cdot m) = (a \otimes 1) \cdot ((1 \otimes b) \cdot m) = ((a \otimes 1)(b \otimes 1)) \cdot m = (a \otimes b) \cdot m$$

Moreover, given M an A-bimodule, we can construct the ring homomorphism

$$\begin{aligned} A \otimes A^{\mathrm{op}} &\to \mathrm{End}_k(M) \\ a \otimes b &\mapsto (m \mapsto amb) \end{aligned}$$

by the universal property of the tensor product via the k-bilinear map

$$\begin{aligned} A \times A^{\mathrm{op}} &\to \mathrm{End}_k(M) \\ (a,b) &\mapsto (m \mapsto amb) \end{aligned}$$

This makes M into a A^e-module.

2. We show that the Hochschild cohomology $HH_n(A,M)$ is the homology given by $M\otimes_{A^e}C_\bullet^{bar}(A)$ where we define the bar complex $C_\bullet^{\mathrm{bar}}(A)$ by $C_n(A)=A^{\otimes n+2}$ for $n\geq -1$ with the differential

$$d_n : C_n(A)\to C_{n-1}(A) : (a_0\otimes ...\otimes a_{n+1}) \to \sum_{i=0}^{n}(-1)^i(a_0\otimes ...\otimes a_ia_{i+1}\otimes ...\otimes a_{n+1}), \quad n\geq 0$$

which coincides with the augmentation map μ when $n=0$. We now endow $C_n(A)$ with an A^e-action :

$$(a\otimes a')\cdot(a_0\otimes a_1\otimes ...\otimes a_{n+1}) = (aa_0\otimes a_1\otimes ...\otimes a_{n+1}a')$$

so that we have the chain complex

$$...\overset{d_{n+1}}{\to} A^{\otimes n+2}\overset{d_n}{\to} A^{\otimes n+1}\overset{d_{n-1}}{\to} ... \overset{d_1}{\to} A^{\otimes 2}\overset{d_0=\mu}{\to} A\to 0$$

Before taking $M\otimes_{A^e}-$ on the whole chain complex, observe that

$$M\otimes_{A^e}C_n(A)\cong M\otimes A^{\otimes n}$$

via

$$\varphi : m\otimes_{A^e}(a_0\otimes a_1\otimes ...\otimes a_{n+1})\mapsto (a_{n+1}ma_0)\otimes a_1\otimes ...\otimes a_n$$

and

$$\psi : m\otimes_{A^e}(1\otimes a_1\otimes\cdots\otimes a_n\otimes 1)\mapsfrom m\otimes a_1\otimes\cdots a_n$$

Taking A away off the chain complex and applying $M\otimes_{A^e}-$ then yields

$$...\overset{b_{n+1}}{\to} M\otimes A^{\otimes n}\overset{b_n}{\to} M\otimes A^{\otimes n-1}\overset{b_{n-1}}{\to} ...\overset{b_2}{\to} M\otimes A\overset{b_1}{\to} M\overset{b_0}{\to} 0$$

with $b_n : M\otimes A^{\otimes n}\to M\otimes A^{\otimes n-1}$ precisely equal to the differential of the Hochschild complex $C_\bullet(A;M)$. Indeed, by the following commutating diagram

$$\begin{tikzcd}
M\otimes_{A^e}C_{n+1}(A) \arrow[r, "\mathrm{id}_M\otimes_{A^e}d_{n+1}"] & M\otimes_{A^e}C_n(A) \arrow[d, "\varphi"] \\
M\otimes A^{\otimes n+1} \arrow[u, "\psi"] \arrow[r, "b_{n+1}"'] & M\otimes A^{\otimes n}
\end{tikzcd}$$

we have $b_n=\varphi\circ(\mathrm{id}_M\otimes_{A^e}d_n)\circ\psi$. We can compute it :

$$b_n(m\otimes a_1\otimes ...\otimes a_n) = \varphi\circ(\mathrm{id}_M\otimes_{A^e}d_n)(m\otimes_{A^e}(1\otimes a_1\otimes ...\otimes a_n\otimes 1))$$

$$=\varphi\Bigg(m\otimes\sum_{i=0}^{n}(-1)^i(1\otimes a_1\otimes ...\otimes a_ia_{i+1}\otimes ...\otimes a_n\otimes 1)\Bigg).$$

Here, treating the occurences $i=0$, $i=n$ and the rest separately yields that is is equal to

$$(m\otimes a_1\otimes ...\otimes a_n)+\sum_{i=1}^{n-1}(-1)^i(m\otimes a_1\otimes ...\otimes a_ia_{i+1}\otimes ...\otimes a_n)+(-1)^n(m\otimes a_1\otimes ...\otimes a_n)$$

Which is precisely what we wanted. The conclusion then comes directly from the two chain complexes being the same (Hochschild's and the tensored one) :

$$HH_n(A;M)=H_n(M\otimes_{A^e}C_\bullet^{\mathrm{bar}}(A))$$

3. We will show $HH^n(A, M)$ is isomorphic to the cohomology of $\mathrm{Hom}_k(A^\bullet, M)$. This follows from the identification

$$\mathrm{Hom}_{A^e}(A^{\otimes n+2}, M) \cong \mathrm{Hom}_k(A^{\otimes n}, M)$$

via

$$f \mapsto \varphi_f, \quad \varphi_f(a_1 \otimes ... \otimes a_n) = f(1 \otimes a_1 \otimes ... \otimes a_n \otimes 1)$$

and

$$\varphi \mapsto f_\varphi, \quad f_\varphi(a_0 \otimes ... \otimes a_{n+1}) = a_0 \cdot \varphi(a_1 \otimes ... \otimes a_n) \cdot a_{n+1}.$$

The differential $\delta^n : \mathrm{Hom}_k(A^{\otimes n}, M) \to \mathrm{Hom}_k(A^{\otimes n+1}, M)$ acts as induced by $\mathrm{Hom}(b_n, M)$ through the above identification :

$$\begin{aligned}(\delta^n\varphi)(a_1, ..., a_{n+1}) &= a_1 \cdot \varphi(a_2, ..., a_{n+1}) \\ &+ \sum_{i=1}^{n} (-1)^i \varphi(a_1, ..., a_i a_{i+1}, ... a_{n+1}) \\ &+ (-1)^{n+1}\varphi(a_1, ..., a_n) \cdot a_{n+1}.\end{aligned}$$

4. By definition,

$$HH^0(A; M) \cong \ker \delta^0.$$

Since

$$\delta^0 : M \to \mathrm{Hom}(A, M), \quad (\delta^0 m)(a) = a \cdot m - m \cdot a$$

we directly obtain

$$HH^0(A; M) \cong \{m \in M : a \cdot m = m \cdot a \quad \forall a \in A\} =: M^A.$$

The group M^A is the group of invariants in the sense that the left and right action of A coincide. We can also see its elements as elements invariant under the "diagonal" subalgebra.

5. By definition,

$$HH^1(A; M) \cong \ker \delta^1 / \mathrm{Im}\, \delta^0.$$

As seen previously,

$$(\delta^1\varphi)(a_1, a_2) = a_1 \cdot \varphi(a_2) - \varphi(a_1 a_2) + \varphi(a_1) \cdot a_2$$

With the knowledge of both δ^1 and δ^0, we obtain

$$HH^1(A; M) \cong \frac{\overbrace{\{\varphi \in \mathrm{Hom}(A, M) : \varphi(ab) = a \cdot \varphi(b) + \varphi(a) \cdot b, \quad \forall a, b \in A\}}^{\mathrm{Der}(A,M):=}}{\underbrace{\{\psi_m \in \mathrm{Hom}(A, M) : \psi_m(a') = a' \cdot m - m \cdot a' : m \in M\}}_{\mathrm{InnDer}(A,M):=}}$$

$$\cong \mathrm{Der(A,M)/InnDer(A,M)}.$$

Where the last isomorphism is a definition between the two upper groups and the two lower one (the colours are here to see much clearer what the meaning of each group is, through the chaotic symbols).